THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2060B Mathematical Analysis II
Homework 9 Suggested Solutions

1. (Exercise 8.2.15 of [BS11]) Let $g_n(x) := nx(1-x)^n$ for $x \in [0,1], n \in \mathbb{N}$. Discuss the convergence of (g_n) and $(\int_0^1 g_n dx)$.

   Solution. Note first that when $x = 0$, $g_n(0) = 0$ for all $n \in \mathbb{N}$. Now let $x \in (0,1]$. Then we employ the ratio test (Theorem 3.2.11 of [BS11]). We have

$$\begin{aligned}\frac{g_{n+1}(x)}{g_n(x)} &= \frac{(n+1)x(1-x)^{n+1}}{nx(1-x)^n} = \frac{(n+1)(1-x)}{n} \\ &= 1 - x + \frac{1}{n}(1-x) \to 1 - x < 1 \text{ as } n \to +\infty\end{aligned}$$

   and hence by the Ratio test, we conclude that $g_n(x) \to 0$ as $n \to +\infty$ for all $x \in (0,1]$.

   For the convergence of $(\int_0^1 g_n(x)dx)$, we show that g_n is uniformly bounded on $[0,1]$ for all $n \in \mathbb{N}$ and use the Bounded Convergence Theorem (Theorem 8.2.5 of [BS11]). In the first derivative test, we have

$$0 = g_n'(x) = n(1-x)^n - n^2x(1-x)^{n-1} \Rightarrow x = \frac{1}{n+1}$$

   and using the second derivative, we have

$$\begin{aligned}g_n''(x) &= -2n^2(1-x)^{n-1} + n^2(n-1)x(1-x)^{n-2} \\ g_n''(1/(n+1)) &= -2n^2\left(\frac{n}{n+1}\right)^{n-1} + n^2(n-1)\left(\frac{1}{n+1}\right)\left(\frac{n}{n+1}\right)^{n-2} \\ &= -(1+n)n\left(\frac{n}{n+1}\right)^{n-1} < 0\end{aligned}$$

   and hence we see that $g_n(x)$ achieves maximum on $[0,1]$ at $x = \frac{1}{n+1}$ with value

$$g_n\left(\frac{1}{n+1}\right) = n\left(\frac{1}{n+1}\right)\left(1 - \frac{1}{n+1}\right)^n = \left(\frac{n}{n+1}\right)^n \leq 1.$$

   Moreover, since $g_n(x) \geq 0$ on $[0,1]$, we see that $|g_n(x)| \leq 1$ for all $n \in \mathbb{N}$ and $x \in [0,1]$ and so by the Bounded Convergence Theorem, we conclude that

$$\lim_{n\to+\infty}\int_0^1 g_n(x)dx = \int_0^1 \lim_{n\to+\infty} g_n(x)dx = \int_0^1 0dx = 0.$$

   ◀

2. (Exercise 8.2.17 of [BS11]) Let $f_n(x) := 1$ for $x \in (0, 1/n)$ and $f_n(x) := 0$ elsewhere in $[0,1]$. Show that (f_n) is a decreasing sequence of discontinuous functions that converges to a continuous limit function, but the convergence is not uniform on $[0,1]$.

Solution. It is clear that each f_n is discontinuous at $x = 1/n$. Observe that

$$f_n(x) - f_{n+1}(x) = \begin{cases} 0, & 0 \le 0 < \frac{1}{n+1} \\ 1, & \frac{1}{n+1} \le x < \frac{1}{n} \\ 0, & \frac{1}{n} \le x \le 1 \end{cases} \Rightarrow f_n(x) - f_{n+1}(x) \ge 0, x \in [0,1]$$

and hence f_n is a decreasing sequence of functions. We show that f_n converges to $f(x) := 0$ for $x \in [0,1]$: Note that $f_n(0) = 0$ for all $n \in \mathbb{N}$. Now let $x_0 \in (0,1]$ be given. Then we can find an $N \in \mathbb{N}$ such that $\frac{1}{N} < x_0$ and hence for all $n \ge N$, $|f_n(x_0)| = 0$. So f_n converges pointwise to the zero function, which is continuous on $[0,1]$, since it is constant.

On the other hand, we show that we can find a sequence $(x_n) \subset [0,1]$ such that $|f_n(x_n) - f(x_n)| \ge 1$ to show that this convergence is not uniform. Let $x_n = \dfrac{1}{2n}$, then we see that since $0 < \dfrac{1}{2n} < \dfrac{1}{n}$, $\left|f_n\left(\frac{1}{2n}\right)\right| = 1$ for all $n \in \mathbb{N}$, as required. ◀

3. (Exercise 8.3.8 of [BS11]) Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f'(x) = f(x)$ for all $x \in \mathbb{R}$. Show that there exists $K \in \mathbb{R}$ such that $f(x) = Ke^x$ for all $x \in \mathbb{R}$.

Solution. We split into two cases. We first consider the case when $f(0) \neq 0$. Then $g(x) := f(x)/f(0)$ is well-defined for $x \in \mathbb{R}$ and we verify the following:

$$g'(x) = \frac{f'(x)}{f(0)} = \frac{f(x)}{f(0)} = g(x),$$
$$g(0) = \frac{f(0)}{f(0)} = 1.$$

Then $g(x)$ satisfies the properties of the function in Theorem 8.3.1 of [BS11] and by the uniqueness theorem (Theorem 8.3.4 of [BS11]), we have that $g(x) = e^x$. Hence, we have that $f(x) = f(0)e^x$ and so we take $K = f(0)$.

The second case is if $f(0) = 0$. Then we will show $f(x) = 0$ using the argument in the proof of Theorem 8.3.4 of [BS11]. By induction, $f^{(n)}(x)$ exists on $\mathbb{R}$ for all $n \in \mathbb{N}$ and equals $f(x)$: the base case is given by the assumption in the question; if $f^{(n)}(x)$ exists on $\mathbb{R}$ and equals $f(x)$, then $f^{(n+1)}(x) = (f^{(n)}(x))' = (f(x))' = f'(x) = f(x)$ and hence $f^{(n+1)}(x)$ exists on $\mathbb{R}$ and equals $f(x)$. Let $x \in \mathbb{R}$ be given. Then $f(t)$ is bounded on $[0,x]$, that is, there is an $M > 0$ such that $|f(t)| \le M$ for all $t \in [0,x]$. Then by Taylor expansion, there is a $c_n \in [0,x]$ such that

$$|f(x)| = \left| f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} + \frac{f^{(n)}(c_n)}{n!}x^n \right|$$
$$= \left| \frac{f^{(n)}(c_n)}{n!}x^n \right| \le \frac{M}{n!}|x|^n$$

since this is true for all $n \in \mathbb{N}$ and $\lim\limits_{n\to+\infty} \dfrac{M}{n!}|x|^n = 0$, we have that $f(x) = 0$. Hence, we can take $K = 0$ in this case. ◀

4. (Exercise 8.3.9 of [BS11]) Let $a_k > 0$ for $k = 1, \ldots, n$ and let $A := (a_1 + \cdots + a_n)/n$ be the arithmetic mean of these numbers. For each k, put $x_k := a_k/A - 1$ in the inequality $1 + x \leq e^x$. Multiply the resulting terms to prove the Arithmetic-Geometric Mean Inequality

$$(a_1 \cdots a_n)^{1/n} \leq \frac{1}{2}(a_1 + \cdots + a_n). \tag{1}$$

Moreover, show that the equality holds in 1 if and only if $a_1 = a_2 = \cdots = a_n$.

Solution. Defining x_k as in the question, and following the suggested argument, we see that

$$1 + x_k \leq e^{x_k} \quad \text{for each } k = 1, \ldots, n.$$

Multiplying all of these inequalities together, we obtain

$$\begin{aligned}\prod_{k=1}^{n}(1 + x_k) &\leq e^{x_1}e^{x_2}\cdots e^{x_k} = \exp\left(\sum_{k=1}^{n} x_k\right)\\ &= \exp\left(\frac{a_1 + a_2 + \cdots + a_n}{A} - n\right) = \exp\left(\frac{nA}{A} - n\right) = e^0 = 1.\end{aligned}$$

The left-hand side simplifies to $\prod_{k=1}^{n}(1 + x_k) = \prod_{k=1}^{n}\frac{a_k}{A}$ and hence

$$\prod_{k=1}^{n}\frac{a_k}{A} \leq 1 \Rightarrow \prod_{k=1}^{n} a_k \leq A^n \Rightarrow \left(\prod_{k=1}^{n} a_k\right)^{1/n} \leq A$$

which is the required Arithmetic-Geometric Mean inequality. Note that equality holds in the product inequality above if and only if both sides of the inequality is 1, that is, $x_k = 0$ for each k. Then this gives $x_k = 1 \Leftrightarrow a_k = A$ for each $k = 1, \ldots, n$, that is, if and only if $a_1 = a_2 = \cdots = a_n = A$. ◀

5. (Exercise 9.1.6 of [BS11]) Find an explicit expression for the nth partial sum of $\sum_{n=2}^{\infty} \ln(1 - 1/n^2)$ to show that this series converges to $-\ln 2$. Is this convergence absolute?

Solution. Let $n \geq 2$ be fixed. We consider the n-th partial sum:

$$s_n = \sum_{k=2}^{n} \ln\left(1 - \frac{1}{k^2}\right) = \sum_{k=2}^{n} \ln\left(\frac{k^2 - 1}{k^2}\right) = \sum_{k=2}^{n} \ln(k-1) + \ln(k+1) - 2\ln(k)$$

using logarithmic identities. We prove using induction that this sum is equal to $-\ln(2) - \ln(n) + \ln(n+1)$ for each n. Note that the base case of $n = 2$ is trivial. Suppose $s_n = -\ln(2) - \ln(n) + \ln(n+1)$ for some n. Then by above,

$$\begin{aligned}s_{n+1} &= s_n + \ln(n) + \ln(n+2) - 2\ln(n+1)\\ &= -\ln(2) - \ln(n) + \ln(n+1) + \ln(n) + \ln(n+2) - 2\ln(n+1)\\ &= -\ln(2) - \ln(n+1) + \ln(n+2)\end{aligned}$$

which is the desired formula for s_{n+1} and hence the explicit formula for the n-th partial sum is

$$s_n = -\ln(2) - \ln(n) + \ln(n+1).$$

We prove that the sum converges to $-\ln(2)$. By the continuity of $\ln(x)$, we have

$$\begin{aligned}\lim_{n\to\infty} s_n &= -\ln(2) + \lim_{n\to\infty} \ln\left(\frac{n+1}{n}\right) = -\ln(2) + \ln\left(\lim_{n\to\infty}\frac{n+1}{n}\right)\\ &= -\ln(2) + \ln(1) = -\ln(2).\end{aligned}$$

Note that for $n \geq 2$, $1 - \frac{1}{n^2} \leq 1$ and hence each summand $\ln\left(1 - \frac{1}{n^2}\right) \leq 0$. Therefore, to show that the series is absolutely convergent, we need to show that $\sum_{n=2}^{\infty}\left|\ln\left(1-\frac{1}{n^2}\right)\right| = \sum_{n=2}^{\infty} -\ln\left(1-\frac{1}{n^2}\right)$ converges. A similar induction argument shows that each partial sum in this new series is simply

$$-s_n = \ln(2) + \ln(n) - \ln(n+1) = \ln(2) + \ln\left(\frac{n}{n+1}\right) \to \ln(2) \text{ as } n \to \infty.$$

Therefore, we conclude that the series is absolutely convergent. ◀

6. (Exercise 9.1.12 of [BS11]) Let $a > 0$. Show that the series $\sum(1+a^n)^{-1}$ is divergent if $0 < a \leq 1$ and is convergent if $a > 1$.

Solution. We make use of the fact that if the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. Taking contrapositive, if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges (in the textbook, this is called the n-th Term Test, Theorem 3.7.3 of [BS11]). When $0 < a < 1$, $a^n \to 0$ as $n \to \infty$ and hence $\frac{1}{1+a^n} \to 1 \neq 0$ as $n \to \infty$ and so by above, the series diverges. Similarly, when $a = 1$, $\frac{1}{1+a^n} = \frac{1}{2}$ and so the series diverges. When $a > 1$, we have

$$\frac{1}{1+a^n} \leq \frac{1}{a^n} = \left(\frac{1}{a}\right)^n, \quad \text{and} \quad \sum_{n=1}^{\infty}\left(\frac{1}{a}\right)^n = \frac{1}{1-1/a}$$

the geometric series with ratio $0 < 1/a < 1$. Hence by the comparison test, $\sum_{n=1}^{\infty}\frac{1}{1-a^n}$ converges in this case. ◀

References

[BS11] Robert G. Bartle and Donald R. Sherbert. *Introduction to Real Analysis, Fourth Edition*. Fourth. University of Illinois, Urbana-Champaign: John Wiley & Sons, Inc., 2011. ISBN: 978-0-471-43331-6.